The Ratio Test for the Convergence of a Series
Part I - A Discrete-Time Case

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In mathematics, the ratio test (also known as the Cauchy ratio test) is a test for the convergence of a series. The ratio test determines if an infinite series converges absolutely or diverges. The ratio test can only tell us that a series converges; it cannot give the value to which the series converges.

In this white paper we will develop the mathematics for the ratio test. To assist us in this endeavor we will use the following hypothetical problem...

Our Hypothetical Problem

Assume that we have the following function of time where the variable n is month number (an integer value) and the variable λ is a rate of decay...

$$f(n) = n\,\lambda^n \tag{1}$$

In Equation (1) above if λ is greater than zero then as n (first half of the equation) goes to infinity the multiplier (second half of the equation) goes to zero. We therefore have the following competing limits...

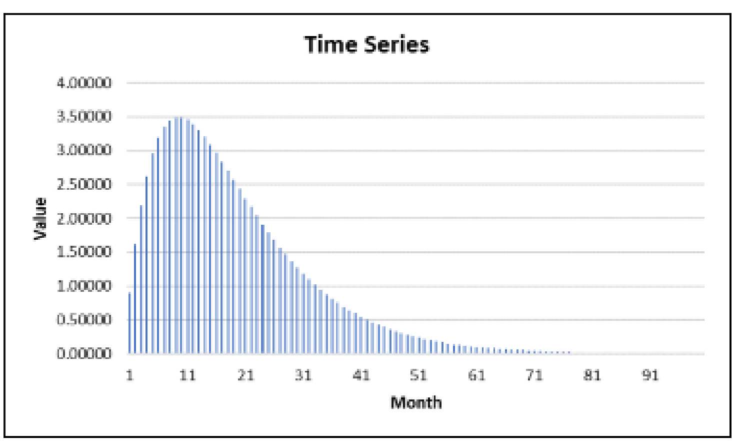
$$\lim_{n\to\infty} n = \infty \text{ ...and... } \lim_{n\to\infty} \lambda^n = 0 \text{ ...when... } \lambda > 0 \tag{2}$$

When n goes to infinity the product of the first term, which goes to infinity, and the second term, which goes to zero, may be infinity or zero. If the product is zero then the series converges. If the product is infinity then the series diverges. Suppose that we want to evaluate the following infinite sum...

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} n\,\lambda^n \tag{3}$$

If the series $f(n)$ converges (product goes to zero) then Equation (3) above has a solution. If the series diverges (product goes to infinity) then Equation (3) above does not have a solution. We will use the ratio test to test for convergence or divergence.

The graph of our time series when $\lambda = 0.90$ is...



Using the product rule the equation for the first derivative of our function in Equation (1) above with respect to time is...

$$\frac{\delta f(n)}{\delta n} = \lambda^n + \lambda^n \ln(\lambda) = \lambda^n \bigg(1 + n\, \ln(\lambda)\bigg) \tag{4}$$

If we set the derivative in Equation (4) above to zero then we can find the equation for the maximum of our function, which is...

$$\lambda^n (1 + n\, \ln(\lambda)) = 0$$
$$1 + n\, \lambda^n = 0$$
$$n = -1/\ln(\lambda) \tag{5}$$

Per the graph of our function when $\lambda = 0.90$ the function increases, reaches a maximum, and then decreases. Using Equation (5) above the maximum value of our function is at $n = -1/\ln(0.90) = 9.5$ years. We can see that after year 9.5 each successive element in the time series converges to zero as month goes to infinity.

Question: Prove that the function in Equation (1) above is convergent.

The Mathematics

We will define the dependent variable $A(n)$ to be the sum of elements in a series where each element in the series is a function of the independent variable n. The equation for $A(n)$ over the inteval $[n = 1, n = \infty]$ is...

$$A(n) = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ... + a_\infty \tag{6}$$

If as n goes to infinity each succesive a_n gets smaller and smaller (i.e. a_n gets closer and closer to zero) then there is a finite solution to the function $A(n)$. If as n goes to infinity each succesive a_n gets larger and larger then there is no finite solution to the function $A(n)$ (i.e. $A(n)$ is infinite).

The common ratio is the amount between each number in a geometric sequence. It is called the common ratio because it is the same to each number, or common, and it also is the ratio between two consecutive numbers in the sequence. The common ratio for the series as defined by Equation (6) above is...

$$\text{Common ratio} = \frac{a_{n+1}}{a_n} \tag{7}$$

The general idea is that if each each element in the series is getting smaller and smaller (i.e. closer and closer to zero) then the series converges and there is a finite solution to the sum of the elements in the series. This occurs if the absolute value of the common ratio as n goes to infinity is less than one. This statement in equation form is...

$$\text{if...}\ \lim_{n\to\infty} \bigg| \text{Common ratio} \bigg| < 1\ \text{...then...}\ A(n)\,\text{converges such that its value is finite} \tag{8}$$

The Solution To Our Hypothetical Problem

Using Equation (8) above the form of the ratio test for our problem is as follows...

$$\text{if}\ \ L = \left| \lim_{n\to\infty} \frac{f(n+1)}{f(n)} \right| < 1\ \text{...then the series...}\ \sum_{n=1}^{\infty} f(n)\ \ \text{converges} \tag{9}$$

Using Equation (1) above the equation for the function $f(n+1)$ is...

$$f(n+1) = (n+1)\,\lambda^{n+1} = (n+1)\,\lambda^n \lambda \tag{10}$$

Using Equations (9) and (10) above the equation for the ratio test is...

$$L = \frac{(n+1)\,\lambda^n \lambda}{n\,\lambda^n} = \frac{n+1}{n}\,\lambda = \bigg(1 + \frac{1}{n}\bigg)\,\lambda \tag{11}$$

If we take the limit of Equation (11) above as time goes to infinity then we get the following equation...

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\lambda=\lambda \ \text{...because...} \ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1 \tag{12}$$

Using Equation (12) above if the following equation holds then the series converges to zero and the integral in Equation (3) has a solution...

$$\text{if } L=\lim_{n\to\infty}\frac{f(n+1)}{f(n)}=\lambda<1 \ \text{ then the series converges} \tag{13}$$

Question: Prove that the function in Equation (1) above is convergent.

Per the ratio test via Equation (13) above the series converges if $\lambda<1$.